UNCERTAINTY IN MEASUREMENTS AND THE PROPAGATION OF ERRORS

Introduction

Every time an engineer takes a measurement, the only certainty is that the measurement is not exact. All measurements have some amount of uncertainty associated with them. For example, if we state that the temperature in a room is 70^{O}F, we do not mean that the temperature is *exactly* 70^{O}F. We probably mean it is close to 70^{O}F. Also, we do not mean that the temperature *everywhere in the room* is 70^{O}F. There will be "hot spots" close to radiators or running electrical equipment. There may be cold spots near windows or doors.

Even "accurate" measurements have some degree of uncertainty. Take, as an example, the problem of weighing yourself on bathroom scales. You may step on the scale once, or 10 times, or 100, or 1000. Let us assume that every time, you record almost the same weight. You may come to the conclusion that since all measurements were very close, you know your weight "accurately". But it is most likely that the bathroom scale has an error and you have very accurately got the wrong answer!

All measurement devices have errors associated with them, and in this module of the course we will learn to identify the most common causes of error. We will then learn how to combine the errors from individual measurements and estimate the error associated with more complex engineering measurements.

The Big Picture

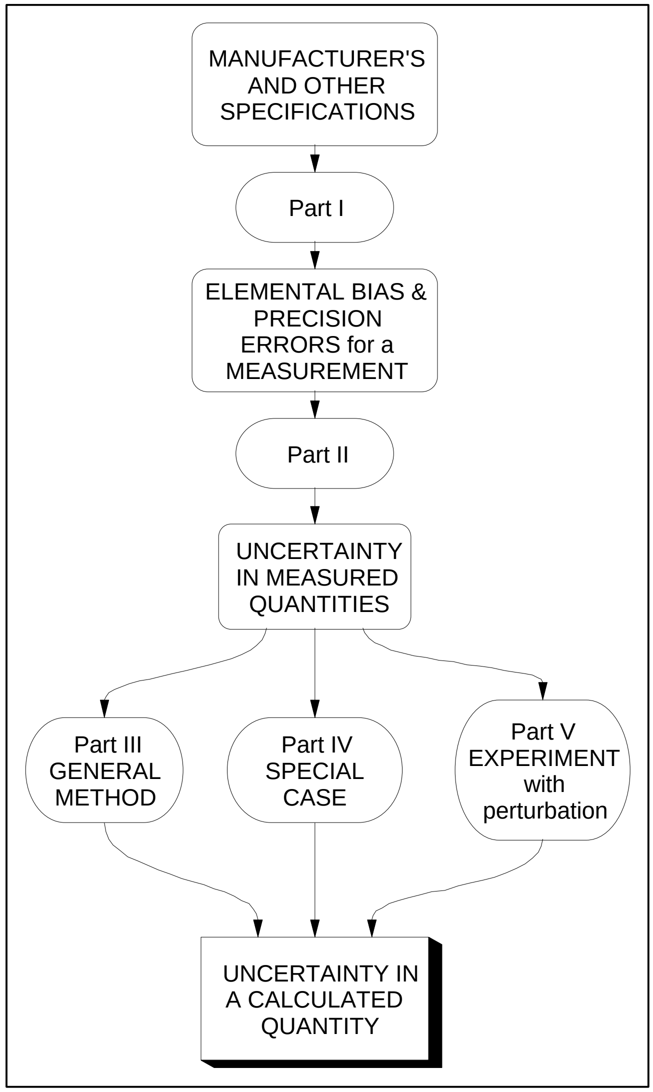
If you measure things, e.g. the width and thickness of a beam, we call those quantities <u>MEASURED QUANTITIES</u>. When you combine several measured quantities together to determine something else, e.g. the second moment of area of the beam, we call that new quantity a <u>CALCULATED QUANTITY</u>.

Width, b = 15 mm MEASURED QUANTITY
Thickness, t = 3 mm MEASURED QUANTITY
Second Moment of Area = $I = \dfrac{bt^3}{12}$ = 33.75 mm^4 CALCULATED QUANTITY

Hot body temperature, T_1 = 600 K MEASURED QUANTITY
Cold body temperature, T_2 = 300 K MEASURED QUANTITY

Radiation Heat Transfer, $q = 5.669\times10^{-8}\times0.8\times(T_1^4 - T_2^4)$ = 5510 W/m^2 CALCULATED QUANTITY

MEASURED QUANTITIES have errors associated with them. We separate those errors into two categories: BIAS ERRORS and PRECISION ERRORS. In **Part I** of this handout, we will learn how to determine the elemental Bias and Precision errors for different measurements based on manufacturer's and other specifications.



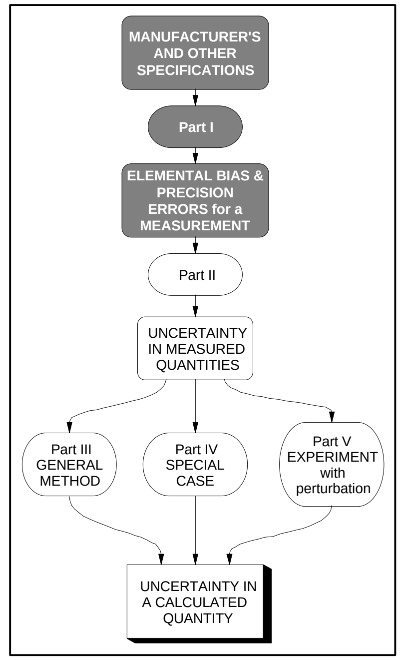
In **Part II** we will learn how to combine these elemental Bias and Precision errors to estimate the uncertainty in a particular measured quantity.

Part III of this handout shows how to combine the uncertainties in measured quantities so that we can estimate the uncertainty in a calculated quantity.

When propagating the uncertainties in measured quantities, there is one special case we have to consider. This special case is widespread in engineering, and knowing when and how to use it can significantly speed up the calculations associated with error propagation. We deal with this special case in **Part IV**.

Finally, **Part V** looks at an empirical approach to error analysis and propagation. This method is closely allied to the theoretical methods in the earlier parts. It is often the case that the equations used to determine a calculated quantity are either not known, or are intractable. This section of the notes introduces a perturbation method that can help estimate some of the errors in this situation.

PART I – Determination Of Elemental Errors From Manufacturer's And Other Specifications

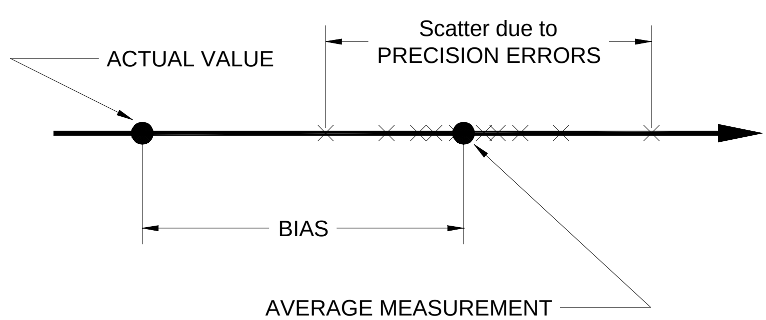


When we use a transducer or measurement system to measure something, we will be involved with a number of different types of error. We normally assign these errors to two different categories: **Bias Errors (B)** and **Precision Errors (P).**

Bias errors are those that are systematic. We can expect the error to be the same each time. For example, if your bathroom scale is set to show 5 lbs too light, you will measure your weight 5 lbs light every time. As a result, bias errors are not obvious. Just because you can get an experiment to give repeatable results, does not mean it is accurate!

Precision errors are those that are random in nature. They vary from time to time, and with enough measurements we can "average out" the error (statistically, we reduce the standard deviation of the mean by increasing the sample size). The precision errors are those that cause your bathroom scale to show a slightly different reading each time you step off and back on.

The following figure demonstrates the difference between bias and precision errors. If the scatter caused by the randomness in the P errors is small compared to the B errors, it is quite possible that the scatter of measurements will never include the actual value. An example of this is your "5 lbs light" bathroom scale if it can repeat readings within ±1 lb.



Before we look at some manufacturer's specifications, let us look at the two different types of error, and see where they come from.

We first look at possible sources of BIAS ERRORS.

Hysteresis: This is the error caused when the reading is different depending on whether the device is being "loaded" or "unloaded" when the measurement is taken. In the balance case, hysteresis is caused by mechanical problems (e.g. friction) in the device. Other types of devices also demonstrate hysteresis. Because the scale will consistently measure the weight too low (for increasing load) or consistently too high (for decreasing load), hysteresis is classified as a bias error.

Common mode voltage: When different voltages are applied to two input terminals of an amplifier, the amplifier will produce an output. Ideally, if the *same* voltage (relative to ground) is applied the terminals the amplifier would produce no output. Common mode voltage error is the error caused by the amplifier actually producing some output under these conditions. It is treated as a bias error.

Installation: An example of this error is when a pitot tube is removed and replaced. If it is not put back in *exactly* the same place the reading will be slightly different.

Nonlinearity (or linearity): This is a common source of a calibration error. It is often assumed that doubling the input to a transducer will double its output. For many transducers this assumption is adequate. However, actual nonlinearity of the transducer will cause the measurement to have a bias error.

Spatial variation: We have already hinted at spatial errors when we said (in the Introduction) that the temperature in a room would vary from place to place. If we were measuring a "hot spot", the thermometer would *always* read too high. Therefore, spatial errors are bias errors.

Loading errors: Imagine putting a cold mercury-in-glass thermometer into a beaker of hot water. Some of the heat will go from the water into the thermometer. As a result, the final temperature you measure will be too low because the water has cooled down a bit. This is an example of a loading error. Many transducers have similar problems. For example, when you use a micrometer to measure the thickness of a piece of paper, the micrometer "squeezes" the paper and the thickness you measure is too small. Loading errors are bias errors.

Zero offset: This is often caused if a device is not "zeroed" properly. That is, when the quantity being measured is zero, the device does not give a zero reading. Zero offset is a bias error.

Sensitivity error: Sensitivity is the measure of how much the output of a transducer varies as the input (the measured quantity) varies. For example, an accelerometer may have a sensitivity of 100 mV/g, indicating that the device will generate a 100 mV signal if the input changes by 1$\times$(acceleration due to gravity). Errors in sensitivity cause bias errors.

Now, let us look at possible sources of PRECISION ERRORS.

Repeatability: Repeatability is the ability of a transducer to give the same output when it is used several times to measure the same thing. Repeatability is a precision error.

Resolution, scale size and quantization errors: Most devices do not give continuous output. Rather, the output is in the form of a series of steps. For example, when you use a simple tape measure to measure a length, you might quote the length to the nearest 1/8-inch. Wire-wound potentiometers are limited to the change in resistance caused by the pick-up moving over a discrete coil. Many digital systems include analog-to-digital conversion, which automatically introduces the "stepped" output, and digital displays are limited to the resolution of the least significant digit. Resolution is treated as a precision error.

Thermal stability: Many systems are sensitive to temperature. For example, the output from a strain gage depends on the resistance of the wire that makes the gage. The wire's resistance depends on temperature, and if a gage is used in a changing temperature environment, the output caused by the temperature change can wrongly be attributed to a changing strain. Amplifiers are typically sensitive to temperature. Mechanical systems can also be sensitive. Consider, for example, a grandfather clock that runs slower in summer than in winter (pendulum gets longer). Thermal stability is treated as a precision error.

Noise: By "noise" we do not (normally) mean acoustic noise. Rather, this usually means the effect on a signal due to electrical interference from surrounding electrical and magnetic fields. Noise is treated as a precision error.

Accuracy - Pandora's Box?

Accuracy is defined as how close the measurement is to the real value. Although we use the term *accuracy*, it is really the *inaccuracy* that is specified. Different manufacturers can have different interpretations of their precise meaning of the term *accuracy*, and in a real-world application you should be careful to ensure you are using the correct definition. However, when accuracy is quoted, it normally includes all the residual B and P errors in the measuring system.

Accuracy is usually quoted as the percentage of full scale. Thus for a balance that can weigh up to 250 lbs with the accuracy quoted as 1% of full scale, the uncertainty is $\pm$2.5 lbs regardless of the reading or divisions on the scale. Since accuracy normally includes both B and P errors, on this course it is assumed accuracy includes all error and uncertainty information.

SUMMARY OF B + P ERRORS

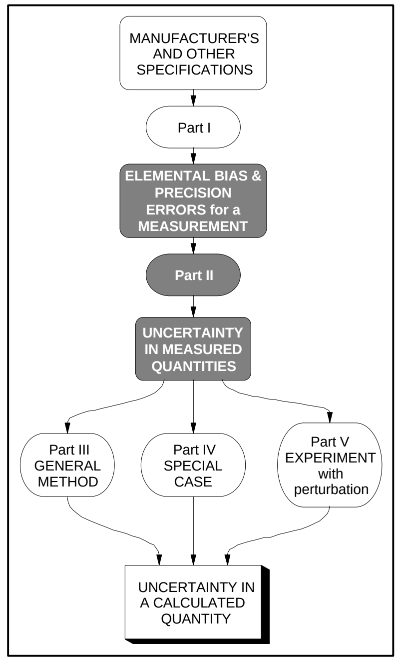
Use this table for all work associated with this course.

Summary of Elemental Errors	
Error	**Error Type**
Hysteresis	Bias
Common Mode Voltage	
Installation	
Linearity	
Spatial Variation	
Loading	
Zero Offset	
Sensitivity	
Repeatability	Precision
Resolution	
Scale Size	
Quantization	
Thermal Stability	
Noise	
Accuracy	Both Precision and Bias

EXAMPLES

This is an advanced copy handout. Examples will be given in class.

PART II – Combining Elemental Errors To Determine The Uncertainty In A Measured Quantity



Bias and precision errors have different effects on a measurement. Therefore, it is highly desirable that we keep track of the different types of error separately before combining them for a final estimate of uncertainty. In this part of the handout, we will first deal with the BIAS errors. We will then inspect the PRECISION errors, and finally we will see how to combine the two types of error.

BIAS ERRORS

The uncertainty due to bias is known as the *bias limit, B*. If a test is repeated under the same conditions, the bias limit remains the same. The bias limit can be estimated if the user has considerable judgment. It is normally determined, though, from calibration tests and comparison with other independent measurements or computer simulations. Since the bias does not change, it cannot be analyzed using statistics. For this reason, instead of using the term confidence level, ANSI/ASME (1986) uses the term *coverage*. Coverage indicates how often the real value is inside the interval estimate. So, for example, if your bathroom scale has a 95% coverage of 5 lbs, we can expect that 95% of the time your real weight is within 5lbs of the reading on the scale.

Combining bias errors. In the previous section, we found that a measurement can be influenced by several different bias errors. If you know or can estimate each separate bias uncertainty, the total bias uncertainty is estimated from:

$$B_x = \left(\sum_i B_i^2 \right)^{1/2}$$

In other words, the total uncertainty due to bias is the RSS (root-sum-of-the-squares) value of all the separate uncertainties due to bias.

PRECISION ERRORS

Precision errors are inherently random in nature, which means they are amenable to a statistical analysis. Let us assume that a test was repeated several times. Using this information, we can calculate the *sample standard deviation* of the measurements, which we call the *Precision Index, S.* During calibration, it is usually possible to conduct different tests in order to estimate the different precision indices, and thus separate out the precision due to, for example, repeatability from that caused by thermal stability.

When we have several precision errors affecting a measurement, we can determine a combined **Precision Index, S_x,** as the RSS of the separate indices:

$$S_x = \left(\sum_i S_i^2 \right)^{1/2}$$

But often we are not too interested in the Precision Index. Rather, we want to know the Precision Limit. This is comparable to the confidence interval we discussed in the separate statistics module in this course. The **Precision Limit, P_X**, for a single measurement is found from the Precision Index as:

$$P_x = tS_x$$

where *t* is the value of Student-t statistic for a chosen level of confidence. In finding the Student-t value, we also need to know the number of degrees of freedom. If the sample size used to determine the individual elemental uncertainties was greater than 30, we can get an approximate value for *t* using the normal distribution (e.g., for 95%, t ≈ 1.96). If *any* sample size is less than 30, ASME/ANSI (1986) suggests the number of degrees of freedom, n, be calculated from the Welch-Satterthwaite formula:

$$n = \frac{\left(\sum_i S_i^2 \right)^2}{\sum_i \left(\frac{S_i^4}{n_i} \right)}$$

In practice, instead of using this detailed equation, we can often use a simpler method of finding the number of degrees of freedom. We saw in the statistics module of this course that for a single variable, the number of degrees of freedom can be calculated as n = n-1. <u>The approximate value for the degrees of freedom for the Precision calculation can be *estimated* by assuming it is the same as the number of degrees of freedom for the uncertainty with the biggest *S/n* ratio.</u>

Let us see how this approximate method compares to the full Welch-Satterthwaite formula. If all the S_i 's are the same, both methods give identically the same result, so let's consider an example where we arbitrarily assign the S_i 's to have the values 1, 2 and 10. Some sample degrees of freedom results are shown in the table:

Sample Sizes	Degrees of Freedom using Welch-Satterthwaite formula	Degrees of Freedom using approximate method
30, 30, 30	>30	29
10, 10, 10	10	9
30, 20, 10	10	9
10, 20, 30	>30	29
2, 4, 6	5	5
6, 4, 2	1	1

The conclusion is that for most engineering purposes, the approximate method is sufficiently accurate for our needs, and unless specifically asked otherwise, you may use the approximate method for this course.

For each Precision Error, calculate S/n with $n = n - 1$

Choose the error with the biggest S/n

The number of degrees of freedom used to find the Student-t value is n = n – 1

COMBINING BIAS AND PRECISION

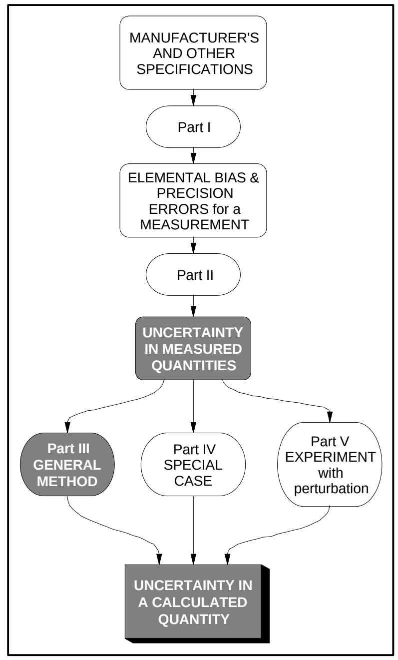
Once we know, separately, the bias and precision uncertainties, we can combine them to find the best estimate of the total uncertainty in a measurement:

$$w_x = \left\{B_x^2 + \left(tS_x\right)^2\right\}^{1/2} = \left\{B_x^2 + P_x^2\right\}^{1/2}$$

Examples

This is an advanced copy handout. Examples will be given in class.

PART III – Propagate Uncertainty In Measured Quantities Into The Uncertainty In A Calculated Quantity – General Procedure



In this Part of the handout, we will look at two different ways of propagating the uncertainty from several measured quantities, into the uncertainty for a calculated quantity. Two examples we will consider are using the tip deflection of a rectangular cross section cantilever to determine the elastic modulus, and calculating the radiation heat flow from a hot body to a cooler body.

GENERAL PRINCIPLES

In general, the quantity we are calculating is a function of several measured quantities. For the cantilever example, the elastic modulus is calculated as:

$$E = \frac{4mgL^3}{ybd^3}$$

where *m* is the mass hung on the end, *L*, *b* and *d* are the length, breadth (width) and depth of the beam respectively, and *y* is the observed tip deflection.

When we say we want to propagate the uncertainty, what we mean is that if there is a small error in, for example, the thickness of the beam, we want to find out how much this effects the measured quantity (the elastic modulus in this example). The error in each measured quantity will effect the result a different amount. We will assume that each measured quantity can be in error by a small amount, *d*. We can find the uncertainty in the elastic modulus from the partial differential equation (note, in this example we have assumed that we know *g* accurately, although this isn't really the case):

$$d_E = d_m \frac{\partial E}{\partial m} + d_L \frac{\partial E}{\partial L} + d_y \frac{\partial E}{\partial y} + d_b \frac{\partial E}{\partial b} + d_d \frac{\partial E}{\partial d}$$

The terms in the partial differential equation have real significance. For example, the term $\frac{\partial E}{\partial m}$ tells us the *sensitivity* of *E* to the mass. In other words, if we change the mass a little bit, how much does this change the calculated value for *E*? We will see in Part V of this handout that we can use experimental methods to measure many of these sensitivities.

MAXIMUM POSSIBLE ERROR. The partial differential equation is only exact if all the d's are infinitesimally small; otherwise it is an approximation. For the approximation, we replace all the d's with the uncertainties in each measured value, w. We also note that the partial terms can be positive or negative, leading to the possibility that one error will apparently "cancel out" another error. To prevent this problem, we only consider the absolute values of the partial terms. Our final result (for this example) is:

$$w_E = \left| w_m \frac{\partial E}{\partial m} \right| + \left| w_L \frac{\partial E}{\partial L} \right| + \left| w_y \frac{\partial E}{\partial y} \right| + \left| w_b \frac{\partial E}{\partial b} \right| + \left| w_d \frac{\partial E}{\partial d} \right|$$

This equation gives us the **MAXIMUM POSSIBLE ERROR** at the level of uncertainty we have chosen.

Cantilever Example. A cantilever is L = 0.80 m long, with an uncertainty of w_L = 0.005 m. It is 35 mm wide and 5 mm thick, with uncertainties of w_b = 0.05 mm and w_d = 0.005 mm respectively. When a mass of 0.5 $\pm$ 0.01 kg was hung on the tip, it was observed that the tip deflection was 11 mm $\pm$ 3%. What are the elastic modulus, and the uncertainty in the modulus? All uncertainties are given to the 95% level of confidence.

Solution: First calculate the elastic modulus. Note the following answer is given far too accurately! After we have completed the propagation of errors, we will know how accurately to quote the answer.

$$E = \frac{4mgL^3}{ybd^3} = \frac{4 \times 0.5 \times 9.81 \times 0.8^3}{0.011 \times 0.035 \times 0.005^3} = 208.736 \text{ GPa}$$

Next, we need to calculate all the partial differential equations. This sounds a lot of work, but really it isn't too bad. Try it!

$$\frac{\partial E}{\partial m} = \frac{4gL^3}{ybd^3} \qquad \left| w_m \frac{\partial E}{\partial m} \right| = \left| 0.01 \times \frac{4 \times 9.81 \times 0.8^3}{0.011 \times 0.035 \times 0.005^3} \right| = 4.17 \text{ GPa}$$

$$\frac{\partial E}{\partial L} = \frac{3 \times 4mgL^2}{ybd^3} \qquad \left| w_L \frac{\partial E}{\partial L} \right| = \left| 0.005 \times \frac{3 \times 4 \times 0.5 \times 9.81 \times 0.8^2}{0.011 \times 0.035 \times 0.005^3} \right| = 3.91 \text{ GPa}$$

$$\frac{\partial E}{\partial y} = \frac{-4mgL^3}{y^2bd^3} \qquad \left| w_y \frac{\partial E}{\partial y} \right| = \left| \frac{0.011 \times 3}{100} \times \frac{-4 \times 0.5 \times 9.81 \times 0.8^3}{0.011^2 \times 0.035 \times 0.005^3} \right| = 6.26 \text{ GPa}$$

$$\frac{\partial E}{\partial b} = \frac{-4mgL^3}{yb^2d^3} \qquad \left| w_b \frac{\partial E}{\partial b} \right| = \left| 0.00005 \times \frac{-4 \times 0.5 \times 9.81 \times 0.8^3}{0.011 \times 0.035^2 \times 0.005^3} \right| = 0.30 \text{ GPa}$$

$$\frac{\partial E}{\partial d} = \frac{-3 \times 4mgL^3}{ybd^4} \qquad \left| w_d \frac{\partial E}{\partial d} \right| = \left| 0.005 \times 10^{-3} \times \frac{-3 \times 4 \times 0.5 \times 9.81 \times 0.8^3}{0.011 \times 0.035 \times 0.005^4} \right| = 0.63 \text{ GPa}$$

We can now find the uncertainty in E as:

$$w_E = \left| w_m \frac{\partial E}{\partial m} \right| + \left| w_L \frac{\partial E}{\partial L} \right| + \left| w_y \frac{\partial E}{\partial y} \right| + \left| w_b \frac{\partial E}{\partial b} \right| + \left| w_d \frac{\partial E}{\partial d} \right|$$
$$= 4.17 + 3.91 + 6.26 + 0.30 + 0.63 = 15.27 \text{ GPa } (4.315\%)$$

Our final solution (with suitable rounding) is therefore: E = 209 ± 15 GPa.

Also, as a matter of note, if we wished to improve the accuracy of the experiment, we should try to improve the measurement that has the highest uncertainty. In this case, it is the tip deflection, which introduced 6.26 GPa of possible error. The "most accurate" measurement in our experiment was the beam width, which introduced a possible error of 0.30 GPa. This is the least problematic measurement in our test, so putting a large effort into increasing the accuracy of this measurement is probably not cost effective.

<u>BEST ESTIMATE OF UNCERTAINTY.</u> Now even though we have estimated the maximum possible uncertainty, this assumes that all the errors in each measurement will become a maximum at the same time. This is most unlikely, and to some extent we can expect that some errors will cancel each other. If we wish to determine the **BEST ESTIMATE OF UNCERTAINTY** we modify the formulation, using a RSS approach, to:

$$w_E = \left\{ \left(w_m \frac{\partial E}{\partial m} \right)^2 + \left(w_L \frac{\partial E}{\partial L} \right)^2 + \left(w_y \frac{\partial E}{\partial y} \right)^2 + \left(w_b \frac{\partial E}{\partial b} \right)^2 + \left(w_d \frac{\partial E}{\partial d} \right)^2 \right\}^{1/2}$$

So for this example we find the best estimate of uncertainty in the elastic modulus as:

$$w_E = \left\{ \left(w_m \frac{\partial E}{\partial m} \right)^2 + \left(w_L \frac{\partial E}{\partial L} \right)^2 + \left(w_y \frac{\partial E}{\partial y} \right)^2 + \left(w_b \frac{\partial E}{\partial b} \right)^2 + \left(w_d \frac{\partial E}{\partial d} \right)^2 \right\}^{1/2}$$
$$= \left\{ 4.17^2 + 3.91^2 + 6.26^2 + 2.98^2 + 0.63^2 \right\}^{1/2} = 8.5 \text{ GPa}$$

Our final solution (with suitable rounding) is now: E = 209 ± 8.5 GPa.

Radiation Heat Transfer Example. In a later engineering course you will measure the radiation heat transfer between two bodies. This transfer is a function of a number and the absolute temperatures of the bodies (in the later course, you will learn more about how the number is derived, but for this error propagation example we do not need to know this level of detail). The radiation heat transfer, q (W/m^2), between two bodies is given by:

$$q = c\left(T_1^4 - T_2^4\right)$$

For a particular pair of bodies, $c = 45.0\times10^{-9} \pm 0.2\times10^{-9}$ W/m^2/K^4, T_1 = 700 K and T_2 = 300 K. In measuring the temperatures, you have a choice of two different thermocouples. Complete with all purchase costs, installation costs and signal conditioning, “regular grade” thermocouples cost \$125 each, and “best grade” thermocouples cost \$200 each. Regular thermocouples can record a maximum temperature of 1300^{O}C with an accuracy ±0.15%. Best grade thermocouples can record a maximum temperature of 1700^{O}C with an accuracy of ±0.5^{O}C. You need one thermocouple for each temperature measurement. Determine the accuracy vs. cost for each possible thermocouple configuration.

Solution. The actual heat transfer is calculated from the equation as 10,440 W/m^2.

Let’s set up the required partial differential equations.

$$\frac{\partial q}{\partial c} = \left(T_1^4 - T_2^4\right) = 232\times10^9$$

$$\frac{\partial q}{\partial T_1} = 4cT_1^3 = 61.74$$

$$\frac{\partial q}{\partial T_2} = -4cT_2^3 = -4.86$$

Now the uncertainties:

$$w_c = 0.2\times10^{-9}\ \text{W/m}^2$$

$$w_{T-REGULAR} = \frac{0.15\times1300}{100} = 1.95^{O}\text{C}$$

$$w_{T-BEST} = 0.5^{O}\text{C}$$

The possible configurations are: 2 × best grade; 1 × best (high temp) and 1 × regular (low temp); 1 × regular (high temp) and 1 × best (low temp); 2 × regular. Let’s consider each configuration in turn, and calculate the best estimate of uncertainty.

2 × best:

$$w_q = \left\{ \left(w_c \frac{\partial q}{\partial c} \right)^2 + \left(w_{T_1} \frac{\partial q}{\partial T_1} \right)^2 + \left(w_{T_2} \frac{\partial q}{\partial T_2} \right)^2 \right\}^{1/2}$$

$$= \left\{ \left(0.2\times10^{-9}\times232\times10^{9}\right)^2 + \left(0.5\times61.74\right)^2 + \left(0.5\times4.86\right)^2 \right\}^{1/2} = 55.78 \text{ W/m}^2$$

1 × best (high) 1 × regular (low):

$$w_q = \left\{ \left(w_c \frac{\partial q}{\partial c} \right)^2 + \left(w_{T_1} \frac{\partial q}{\partial T_1} \right)^2 + \left(w_{T_2} \frac{\partial q}{\partial T_2} \right)^2 \right\}^{1/2}$$

$$= \left\{ \left(0.2\times10^{-9}\times232\times10^{9}\right)^2 + \left(0.5\times61.74\right)^2 + \left(1.95\times4.86\right)^2 \right\}^{1/2} = 56.53 \text{ W/m}^2$$

1 × best (low) 1 × regular (high):

$$w_q = \left\{ \left(w_c \frac{\partial q}{\partial c} \right)^2 + \left(w_{T_1} \frac{\partial q}{\partial T_1} \right)^2 + \left(w_{T_2} \frac{\partial q}{\partial T_2} \right)^2 \right\}^{1/2}$$

$$= \left\{ \left(0.2\times10^{-9}\times232\times10^{9}\right)^2 + \left(1.95\times61.74\right)^2 + \left(0.5\times4.86\right)^2 \right\}^{1/2} = 129.0 \text{ W/m}^2$$

2 × regular:

$$w_q = \left\{ \left(w_c \frac{\partial q}{\partial c} \right)^2 + \left(w_{T_1} \frac{\partial q}{\partial T_1} \right)^2 + \left(w_{T_2} \frac{\partial q}{\partial T_2} \right)^2 \right\}^{1/2}$$

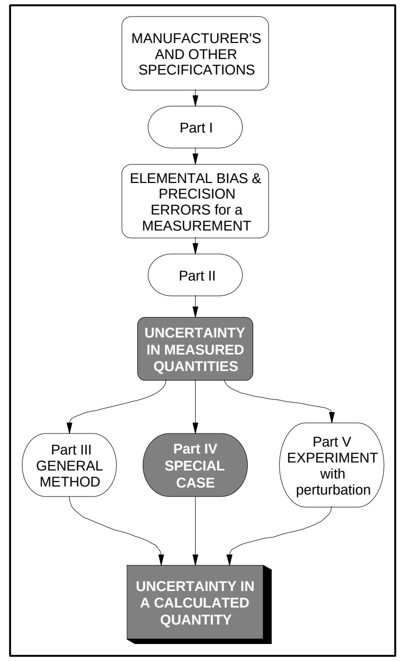
$$= \left\{ \left(0.2\times10^{-9}\times232\times10^{9}\right)^2 + \left(1.95\times61.74\right)^2 + \left(1.95\times4.86\right)^2 \right\}^{1/2} = 129.4 \text{ W/m}^2$$

Summary:

Radiation Heat Transfer, q = 10,440 W/m^2				
High Temp. Thermocouple	Low Temp. Thermocouple	Cost	Uncertainty in q (W/m^2)	Uncertainty in q (%)
Best	Best	$400	55.8	0.53
Best	Regular	$325	56.5	0.54
Regular	Best	$325	129.0	1.24
Regular	Regular	$250	129.4	1.24

Conclusion: Changing the low temperature thermocouple from “best” to “regular” makes very little difference to the final uncertainty. However, having a “best” thermocouple for the high temperature sensor makes a big difference (the uncertainty more than doubles if you use a regular grade thermocouple). The optimum solution is therefore probably to have a best grade thermocouple measuring the high temperature, and a regular grade thermocouple for the low temperature.

PART IV – Propagate Uncertainty In Measured Quantities Into The Uncertainty In A Calculated Quantity – An Important Special Case



In a very large number of engineering situations, the calculated quantity is determined from an equation that is <u>*solely the product of the measured quantities.*</u> There are no additions or subtractions in the equation. There are no functions like sine. In these situations, we can use the work presented in this Part to get a quick way of determining the uncertainty in the calculated quantity.

This "special case" approach gives exactly the same results as using the partial differential equation approach from Part III. But note that the method presented in this Part IV CANNOT BE USED if the equation is not one of the "special case" equations. <u>Therefore, if you are in any doubt, use the full partial differential methods from the previous section.</u>

As examples, the following table shows how some quantities are calculated from other measured quantities and whether each equation is included in this "special case."

Calculated Quantity	Equation	Special Case?	Why not?
E	$E = \frac{4mgL^3}{ybd^3}$	Yes	
q	$q = c\left(T_1^4 - T_2^4\right)$	No	Can't have subtraction
f	$f = \frac{nc^{1/2}}{2L}$	Yes	
h	$h = \sqrt{x^2 + y^2}$	No	Can't have the addition
s	$s = \frac{P}{A}$	Yes	
e	$e = \frac{\left(L - L_O\right)}{L_O}$	No	Can't have subtraction
R	$R = x\sin\left(q\right)$	No	Can't have functions (sine in this case)
f_n	$f_n = \frac{c^2}{2p}\sqrt{\frac{EI}{r A_x L^4}}$	Yes	

For an equation to be considered as the special case, the calculated quantity, C, must be calculated <u>*solely as the product of the measured quantities.*</u>

In general, C must be determined from the measured quantities, x_1, x_2, etc, with an equation of the form:

$$C = x_1^a x_2^b x_3^c \ \ldots$$

In this special case only, the best estimate of the relative uncertainty in C can be determined from:

$$\frac{w_C}{C} = \left\{ \left(a\frac{w_1}{x_1} \right)^2 + \left(b\frac{w_2}{x_2} \right)^2 + \left(c\frac{w_3}{x_3} \right)^2 + \ \ldots \right\}^{1/2}$$

Let’s go back to the example where we used a cantilever to determine the elastic modulus of a material.

$$E = \frac{4mgL^3}{ybd^3} = 4gm^1 L^3 y^{-1} b^{-1} d^{-3}$$

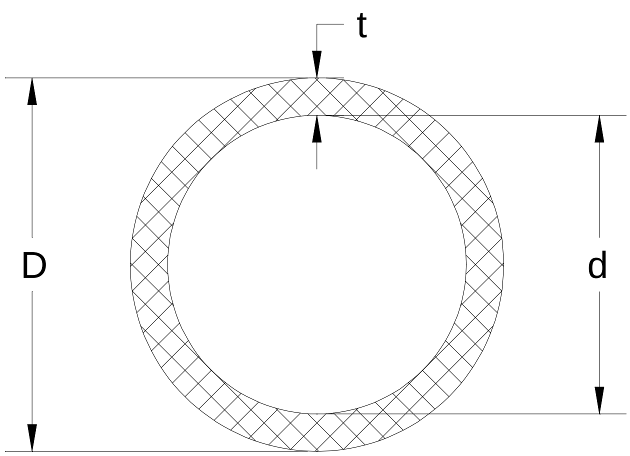
This equation is included as a special case. Therefore, we can estimate the best estimate of the relative uncertainty in E as:

$$\begin{aligned}\frac{w_E}{E} &= \left\{ \left(1\times\frac{w_m}{m} \right)^2 + \left(3\times\frac{w_L}{L} \right)^2 + \left(-1\times\frac{w_y}{y} \right)^2 + \left(-1\times\frac{w_b}{b} \right)^2 + \left(-3\times\frac{w_d}{d} \right)^2 \right\}^{1/2} \\ &= \left\{ \left(1\times\frac{0.01}{0.5} \right)^2 + \left(3\times\frac{0.005}{0.8} \right)^2 + \left(-1\times\frac{3}{100} \right)^2 + \left(-1\times\frac{0.05}{35} \right)^2 + \left(-3\times\frac{0.005}{5} \right)^2 \right\}^{1/2} \\ &= 0.04077 = 4.077\%\end{aligned}$$

Hence w_E = 0.04077 $\times$ E = 8.5 GPa. This is identical to the result we found using the full partial differential equation method. This “special case” approach is very useful. It enables us to quickly use the relative (e.g. percentage) uncertainties of the measurands (measured quantities) to determine the relative uncertainty in the calculated quantity.

Also, using this approach clearly shows us that when a calculation raises a measured quantity to a power (e.g., L^3) the propagation of the error in that quantity has (in this case) three times the effect of a measured quantity that is not raised to a power (e.g., the mass term, m).

AN IMPORTANT NOTE ON INDEPENDENCE. Although we have not mentioned it earlier in this handout, for the propagation of errors to be successful it is important that each measured quantity is independent of all the other measured quantities. As an example, consider measuring the cross-sectional area of a ring:



There are three possible measurements we could take. The outer diameter, *D,* the inner diameter, *d*, and the wall thickness, *t*. Let us use the example with *D* = 2.5 ins, *d* = 2 inches, and *t* = 0.25 ins.

Now let us assume that our two independent measurements were the outside diameter, *D*, and the wall thickness, *t*, both measured with an uncertainty of *w* = 0.01 ins.

The cross-sectional area of the ring is:

$$A = \frac{\pi}{4}\left(D^2 - (D-2t)^2\right) = \frac{\pi}{4}\left(Dt - t^2\right) = 1.767 \text{ in}^2$$

Now let us look at the propagation of errors. Calculating the partial differential equations (since this example is not a special case) we get:

$$w_A = \left\{\left(w_D \frac{\partial A}{\partial D}\right)^2 + \left(w_t \frac{\partial A}{\partial t}\right)^2\right\}^{1/2} = \left\{\left(0.01\frac{\pi t}{4}\right)^2 + \left(0.01\times\frac{\pi\,(D-2t)}{4}\right)^2\right\}^{1/2} = 0.01583 \text{ in}^2$$

Hence, the area is correctly given as: *A* = 1.767 ± 0.016 in^2.

Now let us assume that by mistake (of course!) you wanted to use the area equation:

$$A = \frac{\pi}{4}\left(D^2 - d^2\right)$$

You quickly calculate (from your measurements) that *d* = *D* – 2*t* = 2.0 inches, and then propagate the errors. But since *d* is a calculated quantity, the propagation has to be in steps:

<u>First, find the uncertainty in *d, w*</u>

First, find the uncertainty in *d*, w_d:

$$w_d = \left\{\left(w_D \frac{\partial d}{\partial D}\right)^2 + \left(w_t \frac{\partial d}{\partial t}\right)^2\right\}^{1/2} = \left\{(0.01\times 1)^2 + (0.01\times(-2))^2\right\}^{1/2} = 0.02236 \text{ in}$$

Next, calculate the uncertainty in area. Note that at this stage, you have INCORRECTLY failed to notice that the two quantities, *D* and *d*, are dependent, since *d* was calculated from *D*, and therefore depends on it. The following error propagation is thus incorrect.

$$A = \frac{p}{4}\left(D^2 - d^2\right)$$

$$w_A = \left\{\left(w_D \frac{\partial A}{\partial D}\right)^2 + \left(w_d \frac{\partial A}{\partial d}\right)^2\right\}^{1/2} = \left\{\left(0.01\frac{2pD}{4}\right)^2 + \left(0.02236\frac{(-2)pd}{4}\right)^2\right\}^{1/2} = 0.0702 \text{ in}^2$$

and we incorrectly decide that: $A = 1.77 \pm 0.07$ in^2.

Another example: For a given beam, its natural frequency of vibration depends on the ratio of second moment of area to beam cross-sectional area:

$$f = (\text{const})\sqrt{\frac{I}{A}}$$

For a rectangular section beam, with $b = 30.0 \pm 0.5$ mm and $d = 4.0 \pm 0.1$ mm, find the relative uncertainty in *f*. Note that $I = \frac{bd^3}{12}$ and $A = bd$.

This is a special case. It would seem, therefore, that the best estimate could be determined as:

$$\frac{w_I}{I} = \left\{\left(1\times\frac{w_b}{b}\right)^2 + \left(3\times\frac{w_d}{d}\right)^2\right\}^{1/2} = \left\{\left(\frac{0.5}{30}\right)^2 + \left(3\times\frac{0.1}{4.0}\right)^2\right\}^{1/2} = 0.0768 = 7.68\%$$

$$\frac{w_A}{A} = \left\{\left(1\times\frac{w_b}{b}\right)^2 + \left(1\times\frac{w_d}{d}\right)^2\right\}^{1/2} = \left\{\left(\frac{0.5}{30}\right)^2 + \left(\frac{0.1}{4.0}\right)^2\right\}^{1/2} = 0.0300 = 3.00\%$$

$$\frac{w_f}{f} = \left\{\left(\frac{w_I}{I}\right)^2 + \left(\frac{w_A}{A}\right)^2\right\}^{1/2} = \left\{(0.0768)^2 + (0.0300)^2\right\}^{1/2} = 0.08245 = 8.245\%$$

However, this approach failed to notice that *I* and *A* are dependent on each other. To propagate the errors correctly, we need to remove the dependency:

$$f = (\text{const})\sqrt{\frac{I}{A}} = (\text{const})\sqrt{\frac{bd^3/12}{bd}} = (\text{const})\sqrt{\frac{d^2}{12}} = \frac{(\text{const})}{\sqrt{12}}d$$

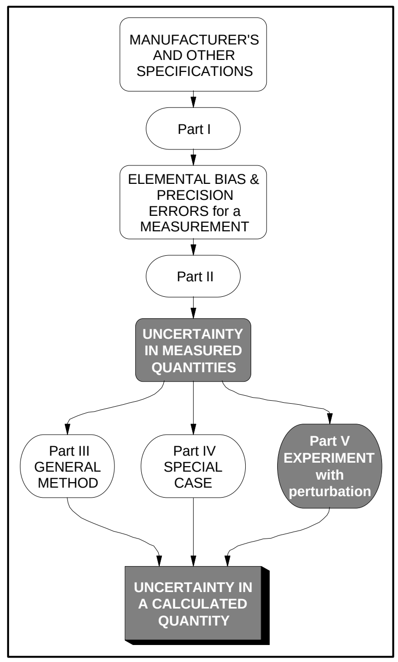
$$\frac{w_f}{f} = \left\{\left(1\times\frac{w_d}{d}\right)^2\right\}^{1/2} = 0.025 = 2.5\%$$

In both these examples, we see that failing to note the dependency of the measurements caused significant mistakes in the estimated uncertainties (0.07 in^2 instead of 0.016 in^2, and 8.2% instead of 2.5%).

In the second example, there is an interesting coincidence. The result (frequency) is completely independent of the beam's width. Thus, it does not matter how accurately (or inaccurately) this measurement is taken. Indeed, the measurement is not even required! This was identified using the correct error analysis, but not noticed in the incorrect approach.

PART V – Propagate Uncertainty In Measured Quantities Into The Uncertainty In A Calculated Quantity – Experiment With Perturbation

It is often the case that the equations used to determine a calculated quantity are either not known, or are intractable. In this **Part V** of the notes we will investigate a perturbation method that can often help estimate some of the errors in this situation.



It should be noted, though, that this perturbation method will ONLY ESTIMATE PRECISION ERRORS. Because bias errors are systematic, they cannot be identified by perturbation.

The experiment can be an actual experiment, or it can be a numerical simulation of an experiment. In both cases, it is necessary that we can apply perturbation methods. The concept of perturbation is relatively easy to grasp.

By *perturbation*, we mean that we make a small change to (perturb) one of the variables in the experiment, and see the effect it has on the output. For example, in an experiment to measure the output power of an internal combustion engine, one of the variables might be the inlet air temperature, T. We may know the uncertainty in air temperature, w_T, (based on how we measured this quantity), but we may not know how that variability affects the engine power, either because we do not know the controlling equations, or they are too complicated to solve analytically. We could set up an experiment where we deliberately increase the inlet temperature by a small amount, and observe the change in engine power.

What we are doing is empirically finding the partial derivative that relates the change in power with change in air temperature. These partial derivatives are the ones we used in Part III to propagate the errors in measured quantities into an uncertainty in a calculated value.

Mathematically, we are saying that, for this example, the engine power, *P*, is some (unknown) function of inlet temperature, *T*.

$$P = f(T)$$

We increase the air temperature by a small amount, *dT*, and observe (measure) the resulting change in power, *dP*. From this information, we can get an approximate numerical value for the rate of change of power with temperature. Suppose a 10^{O}C change in air temperature caused a 2 HP change in power. We get:

$$\frac{\partial P}{\partial T} \approx \frac{dP}{dT} = \frac{2}{10} = 0.20 \text{ HP}/^{O}C$$

Let's assume that the uncertainty in temperature, w_T, is $\pm 3^{O}$C. We can now estimate the uncertainty this causes in power:

$$w_P = w_T \frac{\partial P}{\partial T} \approx w_T \frac{dP}{dT} = 3 \times 0.20 = 0.6 \text{ HP}$$

For a more complete uncertainty analysis we should vary (perturb) every single measured quantity separately, and observe the effect on the calculated property. We would then combine these uncertainties using the methods in Part III, to estimate the overall uncertainty in the final calculated quantity. However, it is very often the case in real-world applications that we wish to see how much a single quantity effects the result.

Consider the water balloon launcher project for this course. You may want to know which parameter is likely to cause the most variability in your achieved range. You could estimate the uncertainty in various parameters (e.g., air density, launch angle, material constants, stretched length, etc.) and use the perturbation methods with your computer programs to estimate the effect of each parameter on the achieved launch range.